Merton's Model in Credit Risk Modelling

Junior BOULLEYS−Associate Partner

Adonis TCHEUGUI−Senior Manager

Hermann NGAMENI−Consultant

Abstract

In this short paper, we first present the Merton's model, a fundamental model for credit risk modeling. Merton's model is a structural model of default in which default occurs at the maturity when the market value of the company's assets fall below a pre-determined threshold defined by liabilities. We then describe the default probability of the company and show some comparative statics analysis.Also, we show some well known strengths and drawbacks of the model. And finally, some models derived Merton's model such as CreditMetrics, KMV model and Credit Risk plus are described.

1 Merton's model

The Merton's model has been introduced in 1974. Its assumptions can be described as follows.

1.1 Assumptions.

- The company is limited and asset follows some stochastic process A_t at date t.
- Company's asset is financed by equity E_t and zero-coupon debt obligation D_t with face value D_T maturing at time $T > t$.
- Market is assumed to be frictionless meaning that the value of the company's asset is equal to the sum of the value of debt obligation and the value of the equity at date t.
- Company cannot pay dividend or issue new debt until the maturity T.
- Default occurs if the company is not able to pay debt holders at maturity.

Under these assumptions, more precisely under assumption 3, we can write down.

$$A_t = E_t + D_t. \tag{1.1}$$

And, there are two possible scenarios at the maturity T.

- The value of the company's assets exceeds the one of liabilities meaning that $A_T > D_T$. In this case, debt holders receive D_T and shareholder $E_T = A_T - D_T$.
- The value of the company's assets is less than the liabilities and the company is not able to meet its obligations. In other words, $A_T < D_T$. Thus, shareholders hand over control of the company to debt holders by exercising the limited liability option. Debt holders thus liquidate the company and distribute the revenues among them.

This can also be summarized by the two following equations.

$$E_T = \max(A_T - D_T, 0). \tag{1.2}$$

And,

$$D_T = \min(A_T, D_T). \tag{1.3}$$

1.2 The default probability

The default occurs in the Merton's model when at maturity T, the value of the company's asset A_T falls below the face value of the debt D_T. Then, the default probability (PD) is computed in this way.

$$PD = pro(A_T < D_T). \tag{1.4}$$

Merton assumes that the company's asset value A_t follows a geometric Brownian motion process, with risk-neutral dynamics given by the following stochastic differential equation.

$$\frac{dA_t}{A_t} = rdt + \sigma_A dw_t. \tag{1.5}$$

where w_t is a standard Brownian motion under risk neutral measure, r denotes the risk-free interest rate, and σ_A is the asset's return volatility. r captures the tendency of the asset's value while the volatility of the asset σ_A is assumed to be constant and captures fluctuations around the tendency.

1.3 A standard Brownian motion

A standard Brownian motion $\{Wt\}_{t \geqslant 0}$ is a stochastic process that satisfies the following properties.

- The function $t \mapsto w_t$ is continuous in t and $w_0 = 0$.
- The process has stationary increments i.e., $w_{t+s} - w_s$ has the same distribution as $w_t - w_0$.
- The process is independent increments i.e., $w_{t_1} - w_{s_1}$ and $w_{t_2} - w_{s_2}$ are independent .
- The increment $w_{t+s} - w_s$ is normaly distributed with mean 0 and variance t.

Applying Îto formula to equation (1.5), we can derive the value of the company's asset at date t as follows:

$$A_t = A_0 \exp\left((r - \sigma_A^2/2)t + \sigma_A w_t\right). \tag{1.6}$$

Then the value of the company's asset at the maturity T is therefore given by following equation.

$$A_T = A_0 \exp\left((r - \sigma_A^2/2)T + \sigma_A w_T\right). \tag{1.7}$$

At date $t = 0$, the value of the equity is given by:

$$E_0 = E(\exp(-rT)\max(A_T - D_T, 0)). \tag{1.8}$$

And the value of the debt is,

$$D_0 = E(\exp(-rT)\min(A_T, D_T)). \tag{1.9}$$

The Black-Sholes formula helps to give another expressions of (1.8) and (1.9).

$$E_0 = A_O \phi(d_1) - D_T \exp(-rT)\phi(d_2). \tag{1.10}$$

And,

$$D_0 = A_O \phi(-d_1) + D_T \exp(-rT)\phi(d_2). \tag{1.11}$$

with $\quad d_1 = \dfrac{\log(A_0/D_T) + (r + \sigma_A^2/2)T}{\sigma_A \sqrt{T}}, \quad d_2 = d_1 - \sigma_A \sqrt{T}.$

ϕ is the cumulative distribution of the standard Gaussian distribution and A_0 the initial value of the company's assets.

The default probability PD given by the equation (1.4) using some manipulations can be written as follows.

$$PD = \phi\left(\frac{\log(D_T/A_0) - (r - \sigma_A^2/2)T}{\sigma_A\sqrt{T}}\right). \tag{1.12}$$

1.3.1 Comparative statics analysis

The default probability satisfies the following conditions.

- It increases with the amount of debt D_T.
- It increases with the volatility σ_A of the company's assets when the value of the debt is less than the inital value of the company's asset.
- It decreases with the initial value of the company's asset A_0.

1.4 Merton's model strengths

- It is easy to compute and provide intuitive results.
- It is able to depict the potential conflict of interest between shareholders and debt holders. The first ones have an interest in the company investing in risky asset project, that increases the volatility of the underlying asset by the way guarantee higher returns, while the second one prefers a less volatile and less risky asset's company value.

1.5 Merton's model flaws

- It does not take into account extrem or rare events by assuming that we are in the Gaussian world.
- The default occurs only at maturity and this is not a realistic since default can occurs before maturity.
- The default corresponds to the liquidation of the company from the market but this is not always the case in most legislation. In US for example, the so-called Chapter 11 allows companies under bankruptcy to try a reorganization, in order to try to become profitable again.

2 Some models derived from Merton's Model

2.1 Moody's KMV model

The KMW model try to overcome some flaws of the Merton's model. KMV model is developed by Moody's Analytics in 2002. In KMV model we model the Expected Default Frequency (EDF) which is the expected probability that a given counterparty default within one year. It is a structural model meaning that the default happens when the market value of the assets of the company, at maturity, is smaller than the value of liabilities.

From equation (1.12) we can write down the following equation.

$$PD = \bar{\phi}(d_2) \tag{2.1}$$

With $\bar{\phi} = 1 - \phi$ the survival function of the Gaussian distribution. D_T is replaced by $\tilde{D}_T$ that reflect the company's liabilities payable with one year. KMV model consider the distant to default defined as

$$d = \frac{A_0 - \tilde{D}_T}{\sigma_A \tilde{D}_T}$$

In the Moody's KMV, the survival function $\bar{\phi}$ is remplaced by a decreasing function estimated using data. This function is denoted ϕ_{KMV} and the Expected Default Frequency becomes

$$EDF = \phi_{KMV}(d) \tag{2.2}$$

2.1.1 KMV model strengths

- The function ϕ_{KMV} probably takes into account extrem events. Therefore it overcomes the gaussian assumption.
- KMV model allows default within one year.

2.2 CreditMetrics model

CreditMetrics is a model introduced by Jp Morgan in 1997. Despite the fact that the model relies on the normality assumption, it presents one interesting characteristics.

- The default threshold is defined using credit ratings not liabilities as in the Merton's model. Thus, it takes into account both the default risk and the credit déterioration which is the second component of the credit risk.
- The model allows for the computation of both the probability of default and the probability of credit deterioration.

2.3 Credit Risk Plus

Credit Risk Plus is introduced in 1997, CR+ is a rather powerful model, which allows for the modeling of complex portfolios of instruments. In this model, default probability is conditionally poisson distributed. It is not a structural model but rather a mixture model. The model is actually used by banks because it introduces more realistic components such as

- Correlation and dependence among defaults.
- Variable default rates.
- Some macroeconomic factors are introduced in order to take into account economic trends and cycles.